Gödel coding on fibrations and geminal categories

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This talk is based on the speaker's master's thesis (Ikeda 2026).

Overview

Ramesh (2023)

- Introduce **introspective theories** and **geminal categories**
- Reveal the categorical structures underlying **Löb's theorem**

Our contributions

- A new, more accessible approach to **geminal categories**
- Establish the **Gödel–Löb axiom** for geminal categories

Outline

1. Background
2. Code structures on fibrations
3. Löb's theorem: $\Box A \vdash A \implies \vdash A$
4. The Gödel–Löb axiom: $\Box(\Box A \to A) \to \Box A$
5. Conclusion

Background

Why categorical approaches to provability logic?

- ‘Mysterious’ appearance of provability logic in CS
  - **Guarded recursion** $\approx$ the strong GL axiom $(\Box A \to A) \to A$ (Nakano 2000)

    $\Box A =$ the type A **guarded** by the later modality
  - **Intensional recursion** $\approx$ the GL axiom $\Box(\Box A \to A) \to \Box A$ (Kavvos 2021)

    $\Box A =$ the type of **codes** of type A, related to metaprogramming

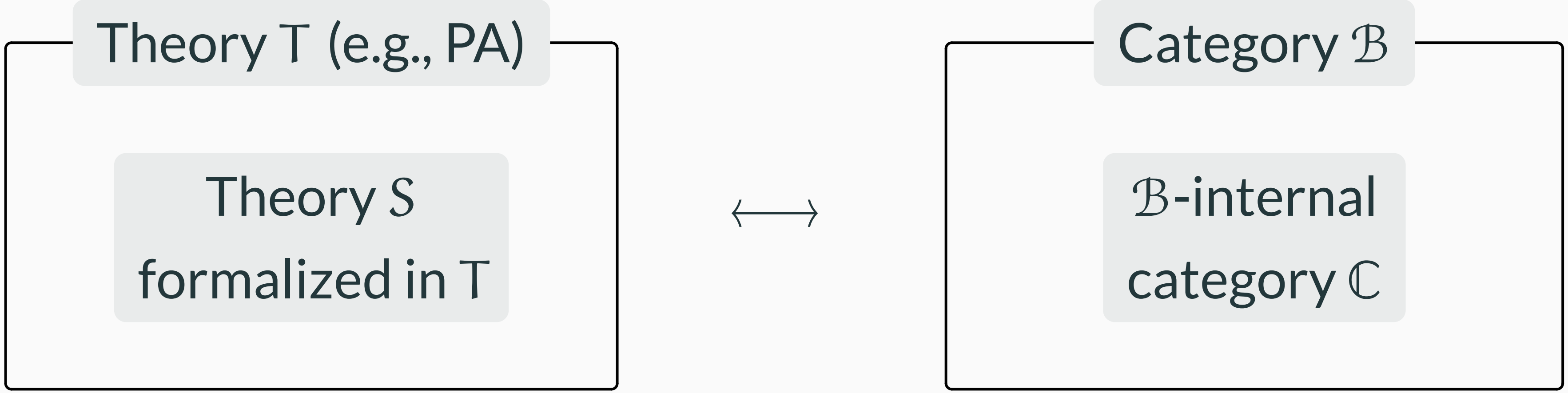
  $\longrightarrow$ Common structures behind provability and certain computation?

- Appropriate categorical structures may refine and generalize GL
  - GL is merely the propositional part of ‘self-internalizing’ systems

Joyal's arithmetic universes

Joyal (1973) *unpublished, see Dijk and Gietelink Oldenziel (2020); Ikeda (2025)

- A categorical interpretation of Gödel's second incompleteness



Theory T (e.g., PA)

Theory $\mathcal{S}$ formalized in T

$\longleftrightarrow$

Category $\mathcal{B}$

$\mathcal{B}$-internal category $\mathbb{C}$

- Introduce **arithmetic universes (AU)** to formalize this idea

Ramesh's work

Ramesh (2023) introduces **introspective theories** and **geminal categories** to unify structures sharing the form of Löb's theorem, $\Box A \vdash A \implies \vdash A$.

Introspective theories / geminal categories arise from:

- Syntactic categories of various logical systems
- Joyal's AU
- Kripke semantics of GL
- Topos of trees $\mathrm{PSh}(\omega)$, a categorical model of guarded recursion

Features of Ramesh’s framework

- Formalize ‘self-internalizing’ aspects via internal categories, à la Joyal
  - **Do not** impose Löb’s theorem / the GL axiom as part of the definitions
  - Rather, Löb’s theorems for them are **proved** in a non-trivial way
  - Explain how (part of) GL arises from ‘self-internalizing’ structures
- High generality
  - Essentially impose only finite-limit structures
  - **Do not** require exponentials, NNOs, …

$\longrightarrow$ May offer general perspectives on meta- and object-level interactions

Limitations and our proposal

Current proofs are intricate and rely heavily on informal ‘internal’ reasoning:

- Limit wider appreciation
- Make it hard to analyze further properties
  - E.g., the GL axiom remains non-trivial, since it seems unclear whether the proof of Löb’s theorem can be internalized in the language of essentially algebraic theories

We introduce a new, accessible approach to **geminal categories:**

- Simplify and reveal the core of the proof of Löb’s theorem
- Enable a completely rigorous proof of the GL axiom
- **Do not** depend on the concept of introspective theories

Code structures on fibrations

The definition of code structures

Definition Let $p : \mathcal{E} \to \mathcal{B}$ be a cloven fibration over a category $\mathcal{B}$ with finite limits. A **code structure** on p consists of $(\ulcorner - \urcorner, \mathsf{app}, \mathsf{code})$ such that:

(i) $$\frac{X \in \mathcal{E}_A}{\ulcorner X \urcorner : 1 \to \ulcorner \mathcal{E}_A \urcorner \ \text{in}\ \mathcal{B}}$$

(ii) $$\frac{f : A \to B \ \text{in}\ \mathcal{B}}{\ulcorner f \urcorner : 1 \to \ulcorner \mathcal{B}(A, B) \urcorner \ \text{in}\ \mathcal{B}}$$

(iii) $\mathsf{app}_{A,B} : \ulcorner \mathcal{E}_B \urcorner \times \ulcorner \mathcal{B}(A, B) \urcorner \to \ulcorner \mathcal{E}_A \urcorner$

s.t. $$\frac{f^*(X) = Y}{\mathsf{app}_{A,B} \circ \langle \ulcorner X \urcorner, \ulcorner f \urcorner \rangle \approx \ulcorner Y \urcorner}$$

(iv) $\mathsf{code}_A : \ulcorner \mathcal{E}_A \urcorner \to \ulcorner \mathcal{B}(1, \ulcorner \mathcal{E}_A \urcorner) \urcorner$

s.t. $$\frac{\ulcorner X \urcorner = x}{\mathsf{code}_A \circ \ulcorner X \urcorner \approx \ulcorner x \urcorner}$$

Here, for $f, g : A \to B$ in $\mathcal{B}$, $f \approx g$ means $\langle f, \mathrm{id} \rangle^* \cong \langle g, \mathrm{id} \rangle^* : \mathcal{E}_{B \times A} \to \mathcal{E}_A$.

Example: Gödel numbering

It is well known that various logical systems (specifications) (Σ, Π) can be organized into the fibration $p : \mathcal{L}(\Sigma, \Pi) \to \mathrm{Cl}(\Sigma)$ (Jacobs 1999).

If (Σ, Π) is countable and contains the natural number type N,
any (usual) **Gödel numbering** on (Σ, Π) induces a code structure on p:

- $\ulcorner \mathcal{E}_A \urcorner = \ulcorner \mathcal{B}(A, B) \urcorner = N \in \mathrm{Cl}(\Sigma)$
- $\ulcorner X \urcorner, \ulcorner f \urcorner : 1 \to N$ are numerals of Gödel numbers of X, f
- app, code are made from corresponding primitive recursive functions

Fixed point theorem for code structures

Theorem Let $p : \mathcal{E} \to \mathcal{B}$ be a fibration equipped with a code structure. If there are $X \in \mathcal{B}$ and $i : \ulcorner \mathcal{E}_X \urcorner \to X$ such that i^* is essentially surjective, for any $F \in \mathcal{E}_{\ulcorner \mathcal{E}_1 \urcorner}$, there exists $A \in \mathcal{E}_1$ such that $(\ulcorner A \urcorner)^* F \cong A$ in $\mathcal{E}_1$.

$$\begin{tikzcd}
\mathcal{E} \arrow[d, "p"'] & \ni & A \arrow[r, "\text{Cart.}"] \arrow[d, mapsto] & F \arrow[d, mapsto] \\
\mathcal{B} & \ni & 1 \arrow[r, "\ulcorner A \urcorner"] & \ulcorner \mathcal{E}_1 \urcorner
\end{tikzcd}$$

For $\mathcal{L}(\Sigma, \Pi) \to \mathrm{Cl}(\Sigma)$, this is exactly the **diagonal lemma,** $\varphi\left(\underline{\ulcorner \chi \urcorner}\right) \leftrightarrow \chi$.

The surjectivity condition is always satisfied by $\mathrm{id} : \ulcorner \mathcal{E}_N \urcorner \to N$.

Löb’s theorem

Code structures, canonically induced

Is it possible to induce code structures more 'canonically' from categorical data, other than from Gödel numbering?

Code structures require some type of **internalizations:**

- $\ulcorner \mathcal{E}_A \urcorner$ internalizes the fiber category $\mathcal{E}_A$,
- $\ulcorner \mathcal{B}(A, B) \urcorner$ internalizes the hom-set $\mathcal{B}(A, B)$, ...

We may use any $\mathcal{B}$-internal category $\mathbb{C}$ as an internalization of $\mathcal{B}$:

- Let $\Gamma\mathbb{C}$ be a category obtained by sending $\mathbb{C}$ along $\Gamma = \mathcal{B}(1, -) : \mathcal{B} \to \mathbf{Set}$
- If we have a functor $F : \mathcal{B} \to \Gamma\mathbb{C}$, then we may set $\ulcorner \mathcal{B}(A, B) \urcorner = \mathbb{C}(FA, FB)$

Pre-geminal categories

Definition A *pre-geminal category* $(\mathcal{B}, \mathbb{C}, F, \mathbb{H})$ consists of:

- A category $\mathcal{B}$ with finite limits*
- A $\mathcal{B}$-internal category $\mathbb{C}$ with finite limits*
- A functor $F : \mathcal{B} \to \Gamma\mathbb{C}$ preserving finite limits*

We write $\Box$ for the endofunctor $\mathbb{C}(1_{\mathbb{C}}, F(-)) : \mathcal{B} \to \mathcal{B}$.

- A $\mathcal{B}$-internal functor $\mathbb{H} : \mathbb{C} \to \Box\mathbb{C}$ such that $(\Gamma\mathbb{H}) \circ F = F^{\#} \circ F$

  Here, $F^{\#} : \Gamma\mathbb{C} \to \Gamma F\mathbb{C}$ is a functor canonically induced by F.

* Choices of limits and their preservation must be *strict*. (The same applies to what follows.)

Code structures from pre-geminal categories

Proposition For any pre-geminal category $(\mathcal{B}, \mathbb{C}, \mathbb{F}, \mathbb{H})$, one can induce code structures on the following in a canonical way:

- Representable fibrations $\mathrm{dom}_A : \mathcal{B}/A \to \mathcal{B}$ for any $A \in \mathcal{B}$
- The codomain fibration $\mathrm{cod} : \mathcal{B}^{\to} \to \mathcal{B}$

Löb's theorem for pre-geminal categories

Theorem Let $(\mathcal{B}, \mathbb{C}, \mathrm{F}, \mathbb{H})$ be a pre-geminal category. For any $A \in \mathcal{B}$ and $f : \Box A \to A$, there exists $a : 1 \to A$ such that $f \circ \mathrm{F}a = a$.

A categorification of **Löb's theorem,** $\Box A \vdash A \implies \vdash A$.

Ramesh has established this result for introspective theories and geminal categories. Our new framework provides:

- More accessible, streamlined proofs (for both structures)
- Slight generalization from geminal to pre-geminal categories

Sketch of the proof

The idea is using the fixed point theorem **twice:**

	Step 1	Step 2
Apply the theorem to:	$\mathrm{cod} : \mathcal{B}^{\to} \to \mathcal{B}$	$\mathrm{dom}_A : \mathcal{B}/A \to \mathcal{B}$
The surjectivity condition is satisfied by:	$(\mathbb{C}/F\mathbb{C}_{\mathrm{mor}})_{\mathrm{ob}} \rightarrowtail \mathbb{C}_{\mathrm{mor}}$	$\mathbb{C}(FB, FA) \cong B$
The theorem yields:	For any $A \in \mathcal{B}$, there is $B \in \mathcal{B}$ s.t. $\mathbb{C}(FB, FA) \cong B$	For any $f : \Box A \to A$, there is $a : 1 \to A$ s.t. $f \circ Fa = a$

This is Ramesh's core insight, which we call the **bootstrapping argument.**

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The Gödel–Löb axiom

Internal pre-geminal categories

The GL axiom $\Box(\Box A \to A) \to \Box A$ is proved by internalizing Löb's theorem.

To perform this, we need the notion of **internal pre-geminal categories:**

pre-geminal category	$\mathcal{A}$-internal pre-geminal category
category $\mathcal{B}$	$\mathcal{A}$-internal category $\mathbb{B}$
$\mathcal{B}$-internal category $\mathbb{C}$	$\Gamma\mathbb{B}$-internal category $\mathbb{C}$
functor $\mathsf{F} : \mathcal{B} \to \Gamma\mathbb{C}$	$\mathcal{A}$-internal functor $\mathbb{F} : \mathbb{B} \to \gamma_{\mathbb{B}}\mathbb{C}$
$\mathcal{B}$-internal functor $\mathbb{H} : \mathbb{C} \to \Box\mathbb{C}$	$\Gamma\mathbb{B}$-internal functor $\mathbb{C} \to \Box_{\mathbb{B}}\mathbb{C}$
$\Gamma\mathbb{H} \circ \mathsf{F} = \mathsf{F}^{\#} \circ \mathsf{F}$	$\gamma_{\mathbb{B}}\mathbb{H} \circ \mathbb{F} = \mathbb{F}^{\#} \circ \mathbb{F}$

Here, $\gamma_{\mathbb{B}} = \mathbb{B}(1_{\mathbb{B}}, -) : \Gamma\mathbb{B} \to \mathcal{A}$ and $\Box_{\mathbb{B}} = \gamma_{\mathbb{C}} \circ \Gamma\mathbb{F} : \Gamma\mathbb{B} \to \Gamma\mathbb{B}$.

Pre-geminal categories almost induce internal ones

Any pre-geminal category $(\mathcal{B}, \mathbb{C}, \mathbb{F}, \mathbb{H})$ itself contains a good candidate for a $\mathcal{B}$-internal pre-geminal category, $(\mathbb{C}, \mathbb{F}\mathbb{C}, \mathbb{H}, \mathbb{F}\mathbb{H})$.

This is not surprising: Recall that the notion of pre-geminal categories arise from requiring internalization of their own structures.

Although, $(\mathbb{C}, \mathbb{F}\mathbb{C}, \mathbb{H}, \mathbb{F}\mathbb{H})$ may fail to be a $\mathcal{B}$-internal pre-geminal category:

- $\mathbb{H}$ need not preserve finite limits
- The required equation, $\Box\mathbb{H} \circ \mathbb{H} = \mathbb{H}^{\#} \circ \mathbb{H}$, need not hold

Geminal categories

If we simply impose these additional conditions on pre-geminal categories, we recover the notion of **geminal categories** by Ramesh:

Definition A **geminal category** is a pre-geminal category $(\mathcal{B}, \mathbb{C}, \mathbb{F}, \mathbb{H})$ such that $\mathbb{H}$ preserves finite limits and $\square\mathbb{H} \circ \mathbb{H} = \mathbb{H}^{\#} \circ \mathbb{H}$ holds.

In fact, it contains the internal version of its own, not only pre-geminal one!

Theorem (Ramesh) Let $(\mathcal{B}, \mathbb{C}, \mathbb{F}, \mathbb{H})$ be a geminal category. Then, $(\mathbb{C}, \mathbb{F}\mathbb{C}, \mathbb{H}, \mathbb{F}\mathbb{H})$ forms a "$\mathcal{B}$-internal geminal category."

The Gödel–Löb axiom for geminal categories

Theorem (I.) Let $(\mathcal{B}, \mathbb{C}, F, \mathbb{H})$ be a geminal category. For any $A \in \mathcal{B}$, there exists $Y_A : \mathbb{C}(F\Box A, FA) \to \Box A$ such that the following commutes:

$$\begin{tikzcd}
\mathbb{C}(F\Box A, FA) \arrow[r, "\langle \mathbb{H}_{1_{\mathbb{C}}, FA} \circ Y_A, \mathrm{id} \rangle"] \arrow[dr, "Y_A"'] & \mathbb{C}(1_{\mathbb{C}}, F\Box A) \times \mathbb{C}(F\Box A, FA) \arrow[d, "\circ_{\mathbb{C}}"'] \\
 & \Box A = \mathbb{C}(1_{\mathbb{C}}, FA)
\end{tikzcd}$$

The type of Y_A corresponds to $\Box(\Box A \to A) \to \Box A$, the GL axiom.

$\mathbb{C}(FA, FB) \in \mathcal{B}$ corresponds to $\Box(A \to B)$ as it internalizes the set $\mathcal{B}(A, B)$.

Comparison to existing models of modal calculi

Kavvos (2020)

- Introduce **dual-context calculi** corresponding to various modal logics
- Present their categorical models in terms of endofunctors $\Box : \mathcal{B} \to \mathcal{B}$

For $(\mathcal{B}, \Box)$ arising from any geminal category, we showed:

- It fully models Kavvos' DK4 ($\approx$ modal logic K4)
- It forms a structure very close to the models of Kavvos' DGL ($\approx$ GL)

There is a subtle mismatch between geminal categories and models of DGL.

This requires further investigation.

Conclusion

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- We present a new, accessible approach to Ramesh's geminal categories. This may contribute to a wider appreciation of the concept.
- The suggestive connection to modal calculi offers a promising explanation for the 'mysterious' appearance of provability logic in CS.
- We anticipate a general structure of "meta- and object-level interactions" — perhaps appropriately called "categorical provability logic."

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