A categorical approach to Gödel's incompleteness via arithmetic universes

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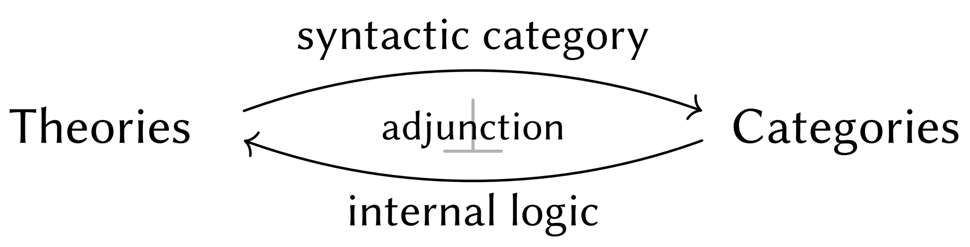
Theory–category correspondence

Categorical logic is a study of the connection between **category theory** and **mathematical logic**. A fundamental observation is that each logical theory can be viewed as a category and vice versa.



This correspondence gives categorical interpretations of logical concepts.
The categorical view can provide:

- **A representation-free ("invariant") way** of studying syntactic theories.
- A unified framework for various semantics.

Joyal's approach to incompleteness

Although **arithmetic** is one of the main topics in mathematical logic, there are only a few studies on its categorical interpretation.

One such research is a categorical approach to Gödel's incompleteness theorems proposed by André Joyal in the 1970s. He introduced categories called **arithmetic universes**, in which arithmetic can be performed sufficiently to satisfy the assumptions of the theorems. Regrettably, none of his works have been published, but some literature mentions his basic ideas [1], [2].

Our research aim is to provide a categorical interpretation of logical phenomena around incompleteness (or, more generally, "theories in other theories") by revisiting and expanding Joyal's idea.

Gödel's incompleteness theorem

We briefly review Gödel's famous (and brilliant) results. For simplicity, we only consider first-order theories in the language $\{0, 1, +, \times, <\}$.

- A **theory** is just a set of logical formulas (called **axioms**).
- A theory T is **consistent** if any contradiction is not derived from T.
- A theory T is **complete** if, for any formula φ, either φ or $\neg\varphi$ is derived from T.
- A theory T is **recursive** if there is an algorithm to decide whether the given formula φ is in T or not.

> **Gödel's first incompleteness theorem.** If a theory T is consistent, recursive, and contains a certain arithmetic, then T is not complete.

The key step of the proof is **to formalize T in T itself.** A rough outline of the proof is as follows:

1. Syntactic entities constituting T (such as terms and formulas) can be encoded as natural numbers (Gödel numbering).
2. Because we assume T contains certain arithmetic, we can express and reason about syntactic properties of T in T itself. In particular, we can express **provability in T** within T itself.
3. Using the diagonal argument, we can create a specific formula G (Gödel sentence), which essentially states **"This statement is not provable in T"**.
4. We can verify that neither G nor $\neg G$ is provable in T.

As we can express provability, the consistency of T is also expressible in T.

> **Gödel's second incompleteness theorem.** If a theory T is consistent, recursive, and contains a certain arithmetic, then T cannot prove the consistency of T itself.

The exact proofs of the theorems depend on the heavy encoding of syntactic entities into natural numbers. A categorical approach would give a "structured" view of incompleteness since it drastically reduces non-canonical encoding.

Arithmetic universe and its construction

> **Definition [3].** An arithmetic universe is a list-arithmetic pretopos.

Through theory–category correspondence, any arithmetic universe has:

- logical operations $\top, \wedge, =, \bot, \vee, \exists$ (but it does not have $\neg, \to, \forall$),
- images of functions,
- quotients over equivalence relations,
- finite lists (particularly **natural numbers**).

There is an arithmetic universe freely generated only from the definition, called **the initial arithmetic universe** $\mathcal{A}_0$.
Joyal provided **a specific construction of** $\mathcal{A}_0$ from primitive recursive functions.

The same construction, internally

The key point of Joyal's idea is:

The construction of $\mathcal{A}_0$ can be performed internally in $\mathcal{A}_0$ itself!

In other words, the above operations are sufficient to do the construction of $\mathcal{A}_0$.
This gives **the internal category** $\mathbb{A}_0$ in the category $\mathcal{A}_0$.
$\mathbb{A}_0$ serves as a categorical analogue of "a theory T formalized in T itself".

Categorical interpretations of logical concepts

In terms of $\mathcal{A}_0$ and $\mathbb{A}_0$, several logical concepts used in the proofs of the incompleteness theorems can be expressed as follows:

Logical concepts	Categorical counterparts
Gödel numbering $\ulcorner - \urcorner$	Unique AU-functor $R : \mathcal{A}_0 \to \mathrm{Ext}(\mathbb{A}_0)$
Provability predicate Pr	Global section functor $\Gamma : \mathrm{Ext}(\mathbb{A}_0) \to \mathcal{A}_0$
Provability modality $\Box$	$\Gamma \circ R : \mathcal{A}_0 \to \mathcal{A}_0$
(Formalized) Σ_1-completeness	(Internal) Freyd cover of $\mathbb{A}_0$
Numeral $\underline{n}$ of a natural number n	$\iota : N \to \mathrm{Hom}_{\mathbb{A}_0}(1_{\mathbb{A}_0}, R(N))$ obtained by the Freyd cover of $\mathbb{A}_0$
Enumeration of formulas (φ_n)	An epimorphism $N \twoheadrightarrow \mathrm{Sub}(R(N))$
$\neg\mathrm{Con}(T)$ ("T is inconsistent")	Equalizer of $R(\mathrm{True})$, $R(\mathrm{False})$

Moreover, we can construct a counterpart of $\neg G$ (the negation of the Gödel sentence) in $\mathcal{A}_0$ by the diagonal argument. Using this, we can prove **both incompleteness theorems for** $\mathcal{A}_0$ in the category-theoretic terms.

Future work

- What is the first-order arithmetic theory that precisely corresponds to $\mathcal{A}_0$?
- How can we deal with **any** recursive theory, not just the above one?
- How can we deal with other incompleteness results, such as Löb's theorem?
  - This might be an uneasy task because arithmetic universes do not have negations or implications. They can contain only Σ_1 formulas [4].
- Internal categories can be viewed as special cases of **fibred categories**.
  Can we interpret "theories in other theories" in terms of fibred categories?

References

This study is a survey that does not contain any mathematical originality (at this stage).
The outline of the categorical proofs of the incompleteness theorems is based on [1].

[1] J. v. Dijk and A. G. Oldenziel, "Gödel incompleteness through Arithmetic Universes after A. Joyal." [Online]. Available: `http://arxiv.org/abs/2004.10482`

[2] A. Morrison, "Reasoning in Arithmetic Universe," 1996.

[3] M. E. Maietti, "Joyal's arithmetic universe as list-arithmetic pretopos," *Theory and Applications of Categories*, vol. 24, Jan. 2010.

[4] L. Ye, "Categorical Structure in Theory of Arithmetic." [Online]. Available: `http://arxiv.org/abs/2304.05477`