

Gödel coding on fibrations and geminal categories

Geminal categories as models of intensional recursion

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Introduction

Gödel–Löb logic GL

- Originally developed for the analysis of **provability** in logical theories
- Kavvos¹: intuitionistic GL models **intensional recursion**
 - cf. a quine (a self-reproducing program)

Question What is the structural connection between logical provability and intensional recursion?

In fact, categorical structures have already been proposed for both sides.

¹ Kavvos (2021), “Intensionality, intensional recursion, and the Gödel–Löb axiom,” *J. Appl. Logics*. 8(8).

The dual-context calculus DGL

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Kavvos¹ introduces the dual-context modal λ -calculus DGL.

DGL provides a formal computational interpretation of GL.

$$\frac{\Delta ; \Delta, z : \Box A \vdash M : A}{\Delta ; \Gamma \vdash \text{fix } z \text{ in box } M : \Box A}$$

(Complementation of variables is omitted.)

Kavvos also gives a categorical semantics for DGL.

We use the following product-based variant.

¹ Kavvos (2020), “Dual-context calculi for modal logic”, *LMCS* 16(3).

A categorical semantics for DGL

Definition An intensional fixpoint category consists of:

- A category $\mathcal{B}$ with finite products
- An endofunctor $\square : \mathcal{B} \rightarrow \mathcal{B}$ preserving finite products
- A natural transformation $\delta : \square \Rightarrow \square \circ \square$ satisfying the coassociative law
- A family of maps $(-)^{\dagger}_{A,B} : \mathcal{B}(B \times \square A, A) \rightarrow \mathcal{B}(\square B, \square A)$ natural in B s.t.

$$\begin{array}{ccccc} \square B & \xrightarrow{\langle \text{id}, \square f^{\dagger} \circ \delta_B \rangle} & \square B \times \square \square A & \xrightarrow{\cong} & \square(B \times \square A) \\ & \searrow f^{\dagger} & & & \downarrow \square f \\ & & & & \square A \end{array}$$

A categorical semantics for DGL

Intensional fixpoint category + Exponentials $\implies$ Model for DGL

Examples:

- Guarded fixpoint category¹ $\implies$ Intensional fixpoint category
(assuming finite-product preservation and naturality)
- Kavvos²: Are there any interesting Gödel–Löb categories that do not arise from guarded fixpoint categories?

¹ Milius and Litak (2017), “Guard your daggers and traces”, *Fundam. Inform.* 150.

² Kavvos (2020), “Dual-context calculi for modal logic”, *LMCS* 16(3).

Geminal categories

Geminal categories

Ramesh¹ introduces **geminal categories** in the context of proposing general structures behind Gödel's incompleteness and Löb's theorem.

Key feature:

- The structure directly formalizes the Gödelian self-formalization through internal categories.
- The Löb property is not part of the definition; it follows as a **theorem**.

¹ Ramesh (2023), *Introspective Theories and Geminal Categories*, Ph.D. dissertation, UC Berkeley.

Geminal categories

Key ingredients of geminal categories

- A category $\mathcal{B}$ with (chosen) finite limits
- An internal category $\mathbb{C}$ in $\mathcal{B}$ with (chosen) internal finite limits
- A functor $F : \mathcal{B} \rightarrow \Gamma\mathbb{C}$ preserving finite limits (strictly)
 - ▶ $\Gamma\mathbb{C}$ is the ordinary category of global objects and morphisms of $\mathbb{C}$

These induce $\square : \mathcal{B} \rightarrow \mathcal{B}$ by

$$\square A := \mathbb{C}(1_{\mathbb{C}}, FA) \in \mathcal{B},$$

the internal hom-objects in $\mathbb{C}$ between global objects $1_{\mathbb{C}}$ and FA .

Examples of geminal categories

Let T be a theory satisfying the usual hypotheses of Gödel's incompleteness theorems (e.g., PA, ZFC, etc.).

One can construct a geminal category such that:

- $\mathcal{B}$ is the ex/reg completion of the syntactic category of T
- $\mathbb{C}$ is the category $\mathcal{B}$ constructed within T , consisting of Gödel numbers
- $F : \mathcal{B} \rightarrow \Gamma\mathbb{C}$ sends a syntactic entity of T to its Gödel number

The induced $\Box : \mathcal{B} \rightarrow \mathcal{B}$ agrees with the **provability predicate** on subterminals (= propositions).

Note that $\mathcal{B}$ has exponentials (if T has connectives $\rightarrow, \forall$).

Result and method

Result

Theorem (The base category of) every geminal category is canonically an intensional fixpoint category.

In particular, if the base is Cartesian closed, it yields a categorical model for the calculus DGL.

Considering the previous example, this result gives:

- A computational content for provability
- A new provability-based model for DGL

It also answers Kavvos' problem positively (pointed out by Ramesh).

Method: Gödel coding on fibrations

We reorganize the theory of geminal categories via abstract Gödel coding:

- **Code structures:** an abstraction of Gödel coding on fibrations
- **Fixed point theorem:** generalizing the diagonal lemma $\varphi(\ulcorner \chi \urcorner) \leftrightarrow \chi$
- A geminal category induces code structures on several typical fibrations
 - e.g., $\text{cod} : \mathcal{B}^{\rightarrow} \rightarrow \mathcal{B}$, $\text{dom}_A : \mathcal{B}/A \rightarrow \mathcal{B}$
- Löb-like properties are proved by applying our fixed point theorem **twice**
 - e.g., first applying to cod , and then to dom_A

This simplifies proofs and makes geminal categories easier to analyze (although the main result also admits direct proofs).

Conclusion

- Any geminal category admits an intensional fixpoint structure.
This provides provability-based models for the calculus DGL.
- We propose a new approach to geminal categories via code structures.
This may provide an accessible framework for the structure.
- The connection between **metamathematics** and **metaprogramming** may suggest a general theory of meta- and object-level interactions
— perhaps appropriately called “categorical provability logic.”

Code structures

Definition Let $p : \mathcal{E} \rightarrow \mathcal{B}$ be a cloven fibration over a category $\mathcal{B}$ with finite limits. A **code structure** on p consists of $(\ulcorner - \urcorner, \text{app}, \text{code})$ such that:

$$(i) \quad \frac{X \in \mathcal{E}_A}{\ulcorner X \urcorner : 1 \rightarrow \ulcorner \mathcal{E}_A \urcorner \text{ in } \mathcal{B}}$$

$$(ii) \quad \frac{f : A \rightarrow B \text{ in } \mathcal{B}}{\ulcorner f \urcorner : 1 \rightarrow \ulcorner \mathcal{B}(A, B) \urcorner \text{ in } \mathcal{B}}$$

$$(iii) \quad \text{app}_{A,B} : \ulcorner \mathcal{E}_B \urcorner \times \ulcorner \mathcal{B}(A, B) \urcorner \rightarrow \ulcorner \mathcal{E}_A \urcorner$$

s.t. $\frac{f^*(X) = Y}{\text{app}_{A,B} \circ \langle \ulcorner X \urcorner, \ulcorner f \urcorner \rangle \approx \ulcorner Y \urcorner}$

$$(iv) \quad \text{code}_A : \ulcorner \mathcal{E}_A \urcorner \rightarrow \ulcorner \mathcal{B}(1, \ulcorner \mathcal{E}_A \urcorner) \urcorner$$

s.t. $\frac{\ulcorner X \urcorner = x}{\text{code}_A \circ \ulcorner X \urcorner \approx \ulcorner x \urcorner}$

Here, for $f, g : A \rightarrow B$ in $\mathcal{B}$, $f \approx g$ means $\langle f, \text{id} \rangle^* \cong \langle g, \text{id} \rangle^* : \mathcal{E}_{B \times A} \rightarrow \mathcal{E}_A$.

Example: Gödel numbering

It is well known that various logical systems (specifications) (Σ, Π) can be organized into the fibration $p : \mathcal{L}(\Sigma, \Pi) \rightarrow Cl(\Sigma)$.

If (Σ, Π) is countable and contains the natural number type $\mathbb{N}$, any (usual) **Gödel numbering** on (Σ, Π) induces a code structure on p :

- $\ulcorner \mathcal{E}_A \urcorner = \ulcorner \mathcal{B}(A, B) \urcorner = N \in Cl(\Sigma)$
- $\ulcorner X \urcorner, \ulcorner f \urcorner : 1 \rightarrow \mathbb{N}$ are numerals of Gödel numbers of X, f
- app, code are made from corresponding primitive recursive functions

Fixed point theorem for code structures

Theorem Let $p : \mathcal{E} \rightarrow \mathcal{B}$ be a fibration equipped with a code structure. If there are $X \in \mathcal{B}$ and $i : \ulcorner \mathcal{E}_X \urcorner \rightarrow X$ such that i^* is essentially surjective, for any $F \in \mathcal{E}_{\ulcorner \mathcal{E}_1 \urcorner}$, there exists $A \in \mathcal{E}_1$ such that $(\ulcorner A \urcorner)^* F \cong A$ in $\mathcal{E}_1$.

$$\begin{array}{ccccc}
 \mathcal{E} & \ni & A & \xrightarrow{\text{Cart.}} & F \\
 p \downarrow & & \downarrow & & \downarrow \\
 \mathcal{B} & \ni & 1 & \xrightarrow{\ulcorner A \urcorner} & \ulcorner \mathcal{E}_1 \urcorner
 \end{array}$$

For $\mathcal{L}(\Sigma, \Pi) \rightarrow \text{Cl}(\Sigma)$, this is exactly the **diagonal lemma**, $\varphi(\ulcorner \chi \urcorner) \leftrightarrow \chi$.

The surjectivity condition is always satisfied by $\text{id} : \ulcorner \mathcal{E}_N \urcorner \rightarrow N$.

The definition of geminal categories

Definition A geminal category $(\mathcal{B}, \mathbb{C}, F, \mathbb{H})$ consists of:

- A category $\mathcal{B}$ with chosen finite limits
 - A $\mathcal{B}$ -internal category $\mathbb{C}$ with chosen finite limits
 - A functor $F : \mathcal{B} \rightarrow \Gamma\mathbb{C}$ strictly preserving finite limits
 - A $\mathcal{B}$ -internal functor $\mathbb{H} : \mathbb{C} \rightarrow \square\mathbb{C}$ strictly preserving finite limits
- satisfying

$$(\Gamma\mathbb{H}) \circ F = F^\# \circ F, \quad \square\mathbb{H} \circ \mathbb{H} = \mathbb{H}^\# \circ \mathbb{H}$$

Here, $F^\# : \Gamma\mathbb{C} \rightarrow \Gamma F\mathbb{C}$, $\mathbb{H}^\# : \square\mathbb{C} \rightarrow \square\square\mathbb{C}$ are functors canonically induced by $F, \mathbb{H}$, respectively.

Code structures from geminal categories

Proposition For any geminal category $(\mathcal{B}, \mathbb{C}, F, \mathbb{H})$, one can induce code structures on the following in a canonical way:

- Representable fibrations $\text{dom}_A : \mathcal{B}/A \rightarrow \mathcal{B}$ for any $A \in \mathcal{B}$
- The codomain fibration $\text{cod} : \mathcal{B}^{\rightarrow} \rightarrow \mathcal{B}$

Löb's theorem for geminal categories

Theorem Let $(\mathcal{B}, \mathbb{C}, F, \mathbb{H})$ be a geminal category. For any $A \in \mathcal{B}$ and $f : \Box A \rightarrow A$, there exists $\alpha : 1 \rightarrow A$ such that $f \circ F\alpha = \alpha$.

This is a categorification of **Löb's theorem**, $\Box A \vdash A \implies \vdash A$.

Sketch of the proof

The idea is to apply the fixed point theorem **twice**:

	Step 1	Step 2
Apply the theorem to:	$\text{cod} : \mathcal{B}^{\rightarrow} \rightarrow \mathcal{B}$	$\text{dom}_A : \mathcal{B}/A \rightarrow \mathcal{B}$
The surjectivity condition is satisfied by:	$(\mathbb{C}/F\mathbb{C}_{\text{mor}})_{\text{ob}} \rightarrowtail \mathbb{C}_{\text{mor}}$	$\mathbb{C}(\text{FB}, \text{FA}) \cong B$
The theorem yields:	For any $A \in \mathcal{B}$, there is $B \in \mathcal{B}$ s.t. $\mathbb{C}(\text{FB}, \text{FA}) \cong B$	For any $f : \square A \rightarrow A$, there is $a : 1 \rightarrow A$ s.t. $f \circ Fa = a$

This is Ramesh's core insight, which we call the **bootstrapping argument**.

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The surjectivity condition is satisfied by:	$(\mathbb{C}/F\mathbb{C}_{\text{mor}})_{\text{ob}} \rightarrowtail \mathbb{C}_{\text{mor}}$	$\mathbb{C}(\text{FB}, \text{FA}) \cong B$
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